Optimal Solution of the Maximum All Request Path Grooming Problem

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***Abstract*— We give an optimal solution to the Maximum All Request Path Grooming (MARPG) problem motivated by a traffic grooming application. The MARPG problem consists in finding the maximum number of connections which can be established in a path of size N, where each arc has a capacity or bandwidth C (grooming factor). We present a greedy algorithm to solve the problem and an explicit formula for the maximum number of requests that can be groomed. In particular, if $C = s(s+1)/2$ and $N > s(s-1)$, an optimal solution is obtained by taking all the requests of smallest length, that is of length 1 to s. However this is not true in general since anomalies can exist. We give a complete analysis and the exact number of such anomalies.**

Keywords : grooming, requests, path, capacity, coloration of interval graphs.

I. Introduction

The **Maximum All Request Path Grooming (MARPG) problem** that we consider in this paper is motivated by traffic grooming in an optical path network, but is of interest by itself. We are given a directed path and a number C (capacity or grooming factor). A request (i, j) is routed via the unique subpath from i to j. The MARPG problem consist in finding the maximum number of simple requests (any request appears at most once) that can be routed (groomed) together such that at most C requests use a given arc of the path. Said otherwise we want that the load of any arc (number of requests whose routing use this arc) does not exceed the capacity C of the arc. One can also formulate the problem as : what is the maximum number of connections that can established in a network where each arc has a capacity (bandwidth) C given, the network being here a path.

The MARPG problem is a particular case of the MRPG (Maximum Request Path Grooming) problem where the set of possible requests is general and not necessarily complete. As noted in [1], [2] there exists a polynomial time algorithm to solve the MRPG problem, and therefore to solve our problem. Indeed the MRPG problem is itself a particular case of the problem of finding a maximum C-colorable subgraph of an interval graph considered in [3], [4]. In this problem we are given a set of n intervals and a number C and we want to find the maximum number of intervals which can be colored with one of the C colors such that two intersecting intervals receive different colors.

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The best known algorithm is given in [3] and has complexity $O(C + n)$. However for our grooming problem we need a closed formula and to the best of our knowledge, this does not exist in the literature. Such a formula is given in Theorem 8.

Let us now explain our motivation. In the original problem of grooming considered in [5] and [6] to each request is associated a route in the optical network and a wavelength ; each request uses at most $1/C$ of the bandwidth or equivalently on a given arc there can be at most C requests with the same wavelength. For a given set of requests the objective is to minimize the number of ADMs (Add Drop Multiplexers) used. This problem has been widely studied in the literature (see the surveys [7], [8], [9], [10]) for various physical networks in particular for the unidirectional ring networks. In [5], [6] the physical network is a dipath. In [5] the problem is proved to be N-P complete for a general set of requests. In [6] the problem is modeled as a graph partition problem as follows : if the set of requests is represented by a graph G, the grooming problem on the path consists in partitioning the edges of G into subgraphs $B_w = (V_w, E_w)$, such that for any arc $(i, i+1)$ of the path there are in each subgraph B_w at most C edges (u, v) with $u \leq i < v$. The objective is to minimize the sum of the number of vertices of the V_w. Here E_w corresponds to the requests with wavelength w and V_w to the number of ADM's used for this wavelength.

To solve the problem, in particular to obtain lower bounds, it is needed to know what is the maximum number of edges $T(C, p)$ that a subgraph B_w with p vertices can contain. This is exactly the MARPG problem for a dipath. Let us be more precise.

First note that for a given set of requests it is easy to compute the load of any arc $(i, i+1)$ of the path; indeed it is the number of requests (u, v) with $u \leq i < v$. In particular, if we have all the possible requests the load of the arc $(i, i+1)$ of the path P_N with N vertices is $(i+1)(n-i-1)$ and so the maximum load of an arc of the path is $\lceil \frac{N^2-\epsilon}{4} \rceil$ where $\epsilon = 1$ if N is odd and 0 if N is even. So the problem is interesting only if C is less than this value otherwise we can groom all the requests.

Now consider the case when the traffic is uniform All to All one, that is the request graph is complete. The lower bound $A(P_N, C)$ for the grooming problem on the path is a solution of the following set of equations where a_p denotes the number of subgraphs of the partition with exactly p vertices (ADMs), each of them satisfying the grooming constraint, and $\varepsilon = 1$

when N is odd and 0 otherwise.

$$A(P_N, C) = \sum_{p=2}^{N} p a_p \quad (1)$$

$$\sum_{p=2}^{N} a_p \geq \left\lceil \frac{N^2 - \varepsilon}{4C} \right\rceil \quad (2)$$

$$\sum_{p=2}^{N} a_p T(C,p) \geq \frac{N(N-1)}{2} \quad (3)$$

Inequality 2 follows from the above computation of the load and inequality 3 from the fact that the number of requests is for the all to all traffic $\frac{N(N-1)}{2}$.

For example, when $C = 2$ it has been proved in [6] that $T(2,p) = \lfloor \frac{3p-3}{2} \rfloor$ and so that $A(P_N, 2) \geq \left\lceil \frac{11N^2-8N-3}{24} \right\rceil$ when N is odd and $A(P_N, 2) \geq \left\lceil \frac{N(N-1)}{3} + \left\lceil \frac{N^2}{8} \right\rceil + \frac{N}{6} \right\rceil$ when N is even. Furthermore a construction attaining this lower bound has been given.

If $C = 1$ the MARPG problem is easy to solve as an optimal solution consists in taking the $N-1$ requests of length 1 and so the maximum is $N-1$. In that case the grooming problem is also solved for any traffic in [6]. In [6] it is also proven that for $C = 2$, the maximum is $\lfloor \frac{3N-3}{2} \rfloor$ by taking all the requests of length 1 and half of that of length 2. By using duality theory, the optimum can be found for $C \leq 6$; in particular for $C = 3$ (respectively $C = 6$) the maximum is obtained by considering all the requests of length 1 and 2 (resp. 1 ,2 and 3), except when $C = 6$ and $N = 5$ where one request of length 4 is also needed.

So it was conjectured the "intuitive fact" that the optimum for the MARPG problem was obtained by taking all the requests of smallest length. However it appears that the conjecture is false (see Section III).

In this article using matroid theory we give a "greedy" algorithm to solve optimally the MARPG problem and determine exactly for any N and C the optimal value of the number of requests that can be groomed.

Note that we present the problem as an oriented one as originally both the requests and the path are directed. But the problem is equivalent to that of considering symmetric requests (a symmetric request being a pair $\{u, v\}$ of nodes communicating) and undirected path.

II. Notations and definitions

Let P_N be the directed path on N vertices $\{0, 1, \ldots, N-1\}$ with the arcs $e_i = (i-1, i), i = 1, \ldots, N-1$. By definition, the request $r_{i,j} = (i, j)$, with $j > i$, loads with load 1 all the arcs of the subpath from vertex i to vertex j. The length or size of a request (i, j) is defined as $s = j - i$. We assume that the set of requests $R(N)$ is composed of all the requests of any length s such that $1 \leq s \leq N-1$. The load of an arc e_i is the number of requests containing e_i that are satisfied (or groomed) together.

The grooming factor C being given, the **Maximum All Request Path Grooming (MARPG) problem** consists in finding the maximum number of requests $T(C, N)$ that can be groomed together such that the load of any arc of P_N is at most C. One can also ask for the requests themselves that are satisfied in an optimal solution.

III. A false conjecture

As said in the introduction intuitively one can think that the maximum is obtained by taking all the requests of smallest length. Furthermore it can be easily proved that it is true on a unidirectional ring for any grooming factor C. For a path P_N, this is clearly true if $C = 1$ as the optimum consists in taking the $N-1$ requests of length 1. So $T(1, N) = N-1$. In [6] this has been also shown for $C = 2$, the maximum being $T(2, N) = \lfloor \frac{3N-3}{2} \rfloor$. This was also proved for $C \leq 6$, except when $C = 6$ and $N = 5$ where we need all requests of length 1, 2, 3 and 4. Hence it was conjectured that it was true for all values of C.

Call $R(s)$ the set of all the requests of size s. Hence we have $R(N) = \bigcup_{s=1}^{N-1} R(s)$. Consider for example the case $C_s = \frac{s(s+1)}{2}$ and $s \leq \frac{N}{2}$. The conjecture can be stated as follows : for $C = C_s$, the maximum number is obtained by taking all the requests of size less than or equal to s. Since the number of requests of size s is equal to $N - s$, the number $T_s(N)$ of requests of size less than or equal to s is $T_s = sN - C_s$. For $C = 3$, we have $T(3, N) = 2N - 3$ and for $C = 6$, $T(3, N) = 3N - 6$ for $N \geq 6$.

However this conjecture is false as can be easily seen from the following example. Let $N = 11$ and $C = 10$, then $s = 4$ and $T_4(11) = 34$. But a better solution exists by deleting from the preceding solution the request $(3, 7)$ of size 4 and adding the two requests of size 5, $(0, 5)$ and $(5, 10)$, which allows to satisfy 35 requests.

Another simple example is given for $s = 6, C_s = 21$ and $N = 16$. We have $T_6(16) = 75$, but we can delete the requests of length 6, $(4, 10)$ and $(5, 11)$, and add the 4 requests of length 7, $(0, 7), (7, 14), (1, 8)$ and $(8, 15)$ leading to a solution with 77 requests. We will see after that these numbers are optimal.

IV. Structuring the request set

There are many ways of enumerating $R(s)$ the set of all the requests of size s. In the following we choose to gather the maximum number of independent requests in a vector (or set) of requests. Recall that $R(s)$ is of cardinality $N - s$ and that each request of size s is of the form $(i, i+s)$ with $0 \leq i \leq N - s - 1$.

For $0 \leq t < s$, let us define a request vector $R_{s,t}$ as the subset of $R(s)$ composed of the requests starting in a vertex $j \equiv t \bmod s$, that is of the form $(t + (h-1)s, t + hs)$. Hence, $R_{s,0}$ is the set of requests $\{(0, s), (s, 2s), ...\}$ and $R_{s,t} = \{(t, t+s), (t+s, t+2s), ...\}$. Note that all the requests of $R_{s,t}$ are independent and that their number is $w(s,t) = \lfloor \frac{N-t-1}{s} \rfloor$.

Lemma 1: For any given s, the union of all $R_{s,t}$ for $0 \leq t \leq s-1$ is equal to $R(s)$.

Proof: The request vectors $R_{s,t}$ are obtained by gathering all the requests $(i, i+s)$ which are equal modulo s. Hence there exist s request vectors. ■

In the remaining part of the paper, we shall prove that there exists a solution to the MARPG problem composed of request vectors; then we will give a greedy algorithm to build such a solution and use it to determine the exact value of the MARPG number.

V. An optimal solution in $R_{s,t}$

In this section, we consider the MARPG problem from the point of view of the requests that will be satisfied in an optimal solution of the problem. The main result is that we can restrict the search for a solution to the set RV of all request vectors $R_{s,t}$ where $1 \leq s \leq N-1$ and $0 \leq t \leq s-1$.

Property 2: The load induced by a set of C request-vectors $R_{s,t}$, with $s \leq s_0$, is C on all the arcs e_j of the path P_N such that $s_0 \leq j \leq N - s_0$.

Proof: A request-vector $R_{s,t}$ loads with load 1 each arc e_j of the path P_N such that $t+1 \leq j \leq t + s.w(s,t)$. But $t+1 \leq s \leq s_0$ and $t + s.w(s,t) \geq N - s \geq N - s_0$, proving the property. ■

Theorem 3: There exists an optimal solution for the MARPG problem consisting of C request-vectors $R_{s,t}$.

Proof: Either there exists an optimal solution consisting of C request-vectors $R_{s,t}$ and we are done. Otherwise, for any optimal solution S, there exists a couple (s,t) such that at least one request of $R_{s,t}$ appears in S and another request of $R_{s,t}$ does not appear in S. Let (s_0, t_0) be the minimum (for the lexicographic order) couple (s,t) with this property. Therefore for any $(s,t) < (s_0,t_0)$, either all the requests of $R_{s,t}$ appear in S or none of them appear in S. Let C_0 be the number of request-vectors $R_{s,t}$, with $(s,t) < (s_0,t_0)$, appearing fully in S.

Consider an optimal solution S_0 such that C_0 is the greatest possible. As S_0 does not consist uniquely of request vectors, we have $C_0 < C$. From S_0 we will build another optimal solution S', such that the request vector R_{s_0,t_0} appears fully in S, and so for this solution we will have $C_0' > C_0$ contradicting the maximality of C_0.

From the definition of (s_0,t_0), it follows that there exist in R_{s_0,t_0} two consecutive requests one appearing in S_0 and one not appearing. We will suppose that the one appearing is before (the case where it is after can be dealt similarly). Let $R_0 = (t_0 + js_0, t_0 + (j+1)s_0)$ be the request which does not appear and $(t_0 + (j-1)s_0, t_0 + js_0)$ be the request appearing. Note that $j \geq 1$ and so $t_0 + js_0 \geq s_0$.

As S_0 is optimal, we cannot add the request R_0 to S_0. Therefore among the arcs covered by the subpath associated to R_0, some of them have load C. Choose the one with the smallest index and call it e^*. It can be written $e^* = e_{t_0+js_0+i_0}$, with $1 \leq i_0 \leq s_0$.

Among the C requests covering e^*, exactly C_0 of them belong to the C_0 request-vectors $R_{s,t}$, with $(s,t) < (s_0,t_0)$, appearing fully in S_0. Therefore there are $C - C_0$ requests covering e^* belonging to some $R_{s,t}$, with $(s,t) > (s_0,t_0)$. Suppose all of them are of the form $(i, i+s)$ with $i < t_0 + js_0$, then all of them also cover the arc $e_{t_0+js_0}$. But this arc is also covered by the C_0 request-vectors $R_{s,t}$, with $(s,t) < (s_0,t_0)$, appearing fully in S_0 (indeed we can apply the property as $t_0 + js_0 \geq s_0$). This arc is also covered by the request $t_0 + (j-1)s_0, t_0 + js_0$ of R_{s_0,t_0} before C_0. So this arc will have a load of $C+1$ which is impossible.

Therefore among the $C - C_0$ requests covering e^* and belonging to some $R_{s,t}$, with $(s,t) > (s_0,t_0)$, at least one request R_1 is of the form $(i, i+s)$, with $s \geq s_0$ and $t_0 + js_0 \leq i \leq t_0 + js_0 + i_0 - 1$. As $s \geq s_0$, the request R_1 covers also all the arcs of R_0 after e^* and by the minimality of e^* the arcs of R_0 before e^* have load at most $C-1$. So, if we delete R_1, all the arcs of R_0 will have a load at most $C-1$ and we can replace R_1 by R_0 without changing the maximum load C obtaining therefore another optimal solution S_1 with one request more than S_0 in R_{s_0,t_0}. Repeating the procedure, we eventually obtain an optimal solution S' containing all the requests of the request vector R_{s_0,t_0} getting the desired contradiction. ■

VI. Optimal greedy algorithm

In this section we shall prove that the request vectors form a weighted matroid. Hence, the associated greedy algorithm will be optimal. Recall (see [11] for details) that a matroid is a pair $M = (S, I)$, where S is a finite nonempty set and I, the independent set, is a nonempty family of subsets of S which satisfies two properties :

1) **Hereditary property** : if $B \in I$ and $A \subset B$, then $A \in I$.
2) **Exchange property** : if $A, B \in I$ and $|A| < |B|$, then there is some element $x \in B - A$ such that $A \cup \{x\} \in I$.

We say that a matroid $M = (S, I)$ is weighted if there is an associated weight function that assigns a strictly positive weight $w(x)$ to each element $x \in S$.

Let N be given, recall that R is the set of all the requests of size not greater than $N-1$. We define RV to be the set of all request vectors $R_{s,t}$, where $1 \leq s \leq N-1$ and $0 \leq t \leq s-1$. From Lemma 1, we deduce that the set of the requests of all the request vectors of size not greater than $N-1$ is equal to R. Hence from the point of view of the requests, R and RV are equal. However they differ from the structural point of view.

Theorem 3 allows us to restrict our search for an optimal solution to RV. For that purpose let us call $P_C(RV)$ the set of all subsets of RV of cardinality at most C, where C is the grooming factor. Hence an element A of $P_C(RV)$ is composed of a set of at most C request vectors $R_{s,t}$ for some values of s and t.

Theorem 4: Given N the size of a directed path and C the grooming factor, the triple $(RV, P_C(RV), w)$ where $1 \leq s \leq N-1$ and $0 \leq t \leq s-1$ and $w(s,t) = \left\lfloor \frac{N-t-1}{s} \right\rfloor$ is a weighted matroid.

Proof: We have to prove that $P_C(RV)$ satisfies the two properties.

1) **Hereditary property** : Let B be a subset of RV; hence B is composed of at most C request vectors. Any subset A of B is composed of less than C request vectors and hence is an element of $P_C(RV)$.
2) **Exchange property** : Let A and B be two subsets of RV such that $|A| < |B|$. Hence there is at least one request vector $R_{s,t}$ in B which does not belong to A. Since $|A| < C$, by adding $R_{s,t}$ to A, we get another set of at most C request vectors, which is clearly in $P_C(RV)$.

∎

In a weighted matroid, the independent subset that has maximum weight is called the optimal subset of the matroid. The main property of a weighted matroid is that a greedy algorithm, considering the elements of S in the decreasing order of their weights, returns an optimal subset (see [11]). We deduce from this property the following theorem.

Theorem 5: Given N the size of a directed path and C the grooming factor, the set of the requests included in the C first request vectors, ordered decreasingly by their weights, is an optimal solution to the MARPG problem.

Proof: From Theorem 3, searching an optimal solution can be restricted to searching in RV. In other words there exists an optimal solution for the MARPG problem which consists of C request vectors.

Since $(RV, P_C(RV), w)$ is a weighted matroid, solving the MARPG problem in RV consists in finding the maximal independent subset with maximal weight where the weight function $w(s,t)$ is the number of requests in $R_{s,t}$. The solution is given by the following greedy algorithm:

Order the set of request vectors by their decreasing weights. The greedy algorithm return the independent set of maximum weight, composed of the first C request vectors and that gives a solution to the MARPG problem. ∎

It is important to understand that the first C request vectors are not composed only with the requests of smallest size. In order to illustrate this statement let us consider the example of Section III. Take $N = 11$, $s = 4$ and $C = 10$. The set of ordered request vectors is the following :

$R_{s,t}$	$R_{1,0}$	$R_{2,0}$	$R_{2,1}$	$R_{3,0}$	$R_{3,1}$	$R_{3,2}$
$w_{s,t}$	10	5	4	3	3	2

$R_{s,t}$	$R_{4,0}$	$R_{4,1}$	$R_{4,2}$	$R_{5,0}$	$R_{4,3}$	$R_{5,1}$	$\dots$
$w_{s,t}$	2	2	2	2	1	1	1

Note that an optimal solution contains the 10 largest request vectors and so we do not take all the 4 request vectors of size 4, as for $R_{4,3}$, $w(4,3) = 1$ but for $R_{5,0}$, $w(5,0) = 2$. Therefore we obtain 35 requests for the maximum number of requests that can be satisfied on a path of size 11 with a grooming factor 10.

Remark that Theorems 4 and 5 work also for a generalization of the MARPG problem, called Maximal Multiple Request Path Grooming (MMRPG) problem. In the MMRPG problem we authorize all $R_{s,t}$ to appear $\lambda_{s,t}$ times, where $\lambda_{s,t}$ is an integer which can be zero, that is the set of requests is $\bigcup \lambda_{s,t} R_{s,t}$. Therefore, an optimal solution is obtained by taking the C first admissible request vectors, ordered decreasingly by their weights.

VII. Computation of $T(C,N)$

We shall show that we can compute exactly the maximum number $T(C,N)$ of requests that can be groomed on a path of size N with a grooming factor C. If we call $RV(C,N)$ the set of C request vectors in an optimal solution of the MARPG problem then we have :

$$T(C,N) \;=\; \sum_{R_{s,t} \in RV(C,N)} w(s,t) \tag{4}$$

Recall that if $C \geq \left\lceil \frac{N^2-\varepsilon}{4} \right\rceil$, where $\epsilon = 1$ when N is odd and 0 otherwise, we can groom all the requests and so $T(C,N) = \frac{N(N-1)}{2}$. Note also that, if $s \geq \frac{N}{2}$, $w(s,t) = 1$; in that case the computation is easy and we delay it to the end of this section. Therefore, let us deal now with the interesting case $s < \frac{N}{2}$, where we will use the greedy algorithm of the preceding section to compute the exact value of $T(C,N)$.

For that we first compute the value for $C = C_s = \frac{s(s+1)}{2}$ and then for any value of C. We have already seen that if the solution $RV(C_s,N)$ contains all (and only them) the request vectors of size at most s, then $T(C_s,N) = T_s = sN - C_s$. But this is not true in general. Hence we shall call **anomalies**, the number of requests that can be satisfied in excess of T_s. This is this number that we have to compute.

First let us note the following property.

Property 6: For fixed value of t, $w(s,t)$ is a decreasing function of s, i.e., for all s, $w(s+1,t) \leq w(s,t)$. For fixed value of s, $w(s,t)$ is a decreasing function of t, i.e.,for all t, $w(s,t+1) \leq w(s,t)$.

Lemma 7: Given N, s such that $1 \leq s < \frac{N}{2}$ and $C_s = \frac{s(s+1)}{2}$, let $N = qs + r$ where $0 \leq r \leq s-1$; $r = aq + \alpha$ where $0 \leq \alpha \leq q-1$ and $s - r = b(q+1) + \beta$ where $0 \leq \beta \leq b-1$. The number of requests of an optimal solution for the MARPG problem is

$$T(C_s,N) \;=\; T_s + \min(A_s, B_s) \tag{5}$$

where $T_s = sN - C_s$, $A_s = ar - \frac{a(a+1)}{2}q$, and $B_s = (b+1)(s-r) - \frac{b(b+1)}{2}(q+1)$.

Proof: Among the s request vectors of $R(s)$, we deduce from the decomposition $N = qs+r$ that there are r vectors of cardinality q (since $w(s,t) = q$) and $s-r$ vectors of cardinality $q-1$. Similarly, among the $s+k$ request vectors of $R(s+k)$, there are $r-kq$ vectors of cardinality q and $s-r+kq$ vectors of cardinality $q-1$. This is true for all values of k such that $r - kq \geq 0$, hence for $k \leq a$. Call A_s the number of request vectors of cardinality q in $R(s+k)$ for $1 \leq k \leq a$. We have : $A_s = ar - \frac{a(a+1)}{2}q$.

Moreover, among the $s-h$ request vectors of $R(s-h)$ there are $s-h-r-hq = s-r-h(q+1)$ vectors of cardinality

$q-1$ for $0 \leq h \leq b$. Call B_s the number of request vectors of cardinality $q-1$ in $R(s-h)$ for $0 \leq h \leq b$. We have : $B_s = (b+1)(s-r) - \frac{b(b+1)}{2}(q+1)$.

By the definition of A_s, we deduce that we will have some anomalies if $A_s \geq 0$. This means that in the request vectors of $R(s+k)$, for $1 \leq k \leq a$, we have some of them with one request more than those in $R(s-h)$, $0 \leq h \leq b$. In the non decreasing order of weights the B_s request vectors of cardinality $q-1$ will appear after the A_s request vectors of cardinality q and so we should replace them. The maximum number of such replacements is $\min(A_s, B_s)$. ■

We can now prove the main result of the paper. For that remark that any positive number C can decomposed into $C = C_s - d$ with $0 \leq d \leq s-1$.

Theorem 8: Given N and C such that $C < \frac{N(N+2)}{8}$ when N is even and $C < \frac{N^2-1}{8}$ when N is odd, let $C = C_s - d$ with $0 \leq d \leq s-1$; $N = qs + r$ where $0 \leq r \leq s-1$; $r = aq + \alpha$ where $0 \leq \alpha \leq a-1$ and $s - r = b(q+1) + \beta$ where $0 \leq \beta \leq b-1$. Then the number of requests of an optimal solution for the MARPG problem is

$$T(C,N) = T_s - dq + \min(A_s + d, B_s) \quad (6)$$

where $T_s = sN - C_s$, $C_s = \frac{s(s+1)}{2}$, $A_s = ar - \frac{a(a+1)}{2}q$ and $B_s = (b+1)(s-r) - \frac{b(b+1)}{2}(q+1)$.

Proof:

Starting from a solution of the MARPG problem with $C = C_s$, we can build a solution for $C = C_s - d$ by removing the last d request vectors from the solution. Among these vectors assume that d_1 are of cardinality q and d_2 of cardinality $q-1$ where $d_1 + d_2 = d$. From the definition of A_s and B_s we deduce that :

$$\begin{aligned} T(C,N) &= T_s - d_1 q - d_2(q-1) \\ &\quad + \min(A_s + d_1, B_s - d_2) \end{aligned} \quad (7)$$

Hence :

$$T(C,N) = T_s - dq + d_2 + \min(A_s + d_1, B_s - d_2) \quad (8)$$

which leads to the formula and concludes the proof. ■

We show in Figure 1 the number of anomalies (gap between $T(C,N)$ and the value of the false conjecture) for grooming factor $C = 192$ and $C = 256$ (effective values for SONET networks).

We now give some complementary properties on the solution of the MARPG problem.

Corollary 9: Let N, s and $C = C_s$ be given, a necessary and sufficient condition for the absence of anomalies is : $(N \bmod s) \leq \lfloor \frac{N}{s} \rfloor$, or equivalently $N = us + v(s+1)$, with u and v positive integers.

Proof: Let $N = qs + r$ with $0 \leq r \leq s-1$, that is $r \equiv N \bmod s$ and $q = \lfloor N/s \rfloor$. From the definition of B_s, it can be easily seen that $B_s \geq s - r > 0$. Hence in order to have no anomalies, a necessary and sufficient condition is that $A_s = 0$. This condition is equivalent to $r \leq q$ which can be also written as $N = (q-r)s + r(s+1)$. ■

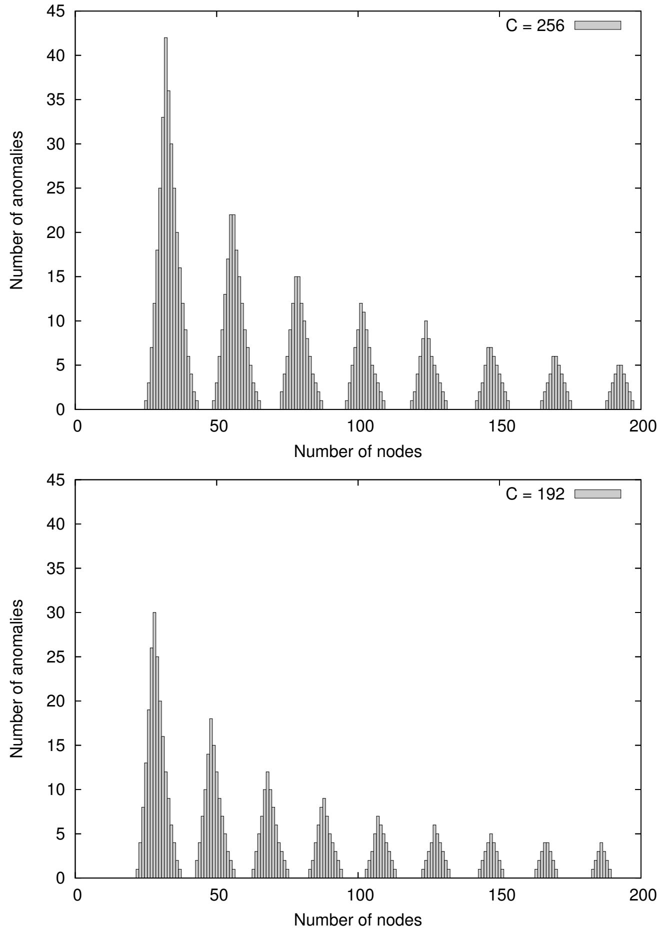

Fig. 1. Number of anomalies for grooming factor $C = 256$ and 192.

Corollary 10: Let s and $C = C_s$ be given. If $N > s(s-1)$, then there is no anomalies and $T(C_s, N) = T_s$.

To be complete, it remains to compute the value of $T(C,N)$ when $s \geq \frac{N}{2}$.

Theorem 11: The number of requests of an optimal solution for the MARPG problem is

- For N even and $\frac{N(N+2)}{8} \leq C \leq \frac{N^2}{4}$

$$T(C,N) = \frac{N(N-2)}{4} + C \quad (9)$$

- For N odd and $\frac{N^2-1}{8} \leq C \leq \frac{N^2-1}{4}$

$$T(C,N) = \frac{N(N-1)}{4} + C \quad (10)$$

Proof: The proof use the fact that for $\frac{N}{2}+1 \leq s \leq N-1$ and $0 \leq t \leq s-1$, $w(s,t) = 1$.

- **Case 1:** N even
  In that case the number of requests vectors of size less or equal to $\frac{N}{2}$ is: $\sum_{s=1}^{\frac{N}{2}} s = \frac{N(N+2)}{8}$. Furthermore, we have $w(s,t) = 1$ for all $\frac{N}{2}+1 \leq s \leq N-1$ and $0 \leq t \leq$

$s-1$. Thus, when $C = \frac{N(N+2)}{8}$, we can build a solution containing all request vectors of size less or equal to $\frac{N}{2}$ and we have no interest to choose requests of greater size. Finally, as the number of requests of size s is $N-s$, we have for $C = \frac{N(N+2)}{8}$, $T(C,N) = \sum_{s=1}^{\frac{N}{2}} N - s = \frac{N(3N-2)}{8}$.

For $C > \frac{N(N+2)}{8}$ we have to add in the optimal solution $C - \frac{N(N+2)}{8}$ of larger size and so $T(C,N) = \frac{N(3N-2)}{8} + C - \frac{N(N+2)}{8} = \frac{N(N-2)}{4} + C$.

- **Case 2:** N odd
  Like in the preceding case, we obtain for $C = \frac{N^2-1}{8}$, $T(C,N) = \frac{(N-1)(3N-1)}{8}$, and for $C > \frac{N^2-1}{8}$, $T(C,N) = \frac{N(N-1)}{4} + C$.

■

VIII. Conclusion

In this article we have completely solved the problem of determining the maximum number of requests which can be groomed in a path with a capacity C on each arc. We have shown furthermore that optimal solutions were obtained with a greedy algorithm. It will be interesting to consider the same problem for other networks in particular to determine networks for which the solution is obtained using request vectors or for which there exists a polynomial algorithm to solve the problem. Note that for unidirectional rings this problem is easy as the solution is given by considering all the requests of smallest length.

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